Efficient and accurate calculations of stability bounds in Hamiltonian systems

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The stability of motion in a typical Hamiltonian system with two degrees of freedom is described by the corresponding fractal diagram and the corresponding critical function. We extend the method of modular smoothing on the fractal diagram by showing that it possesses the same transformation properties as the critical function. This enables one to calculate the fractal diagram at all points and the critical function knowing only a few periodic orbits with the smallest periods.

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I. INTRODUCTION

The problem of efficient and accurate calculations of fractal diagrams and critical functions is important for a successful description of transport in Hamiltonian systems [1]. A fractal diagram, denoted by $k_c(m/n)$, for a one-parameter family of area-preserving twist maps of a cylinder completely describes the stability of rotational periodic orbits. It is a complicated fractal function of rational frequencies of the rotational periodic orbits. The critical function, denoted by $K(\nu)$, describes a relation between values of the parameter, at which invariant rotational tori are destroyed, and irrational frequencies of quasiperiodic orbits on such tori. It is also a complicated fractal function. The main result presented in this paper, is that the knowledge of the fractal diagram at only a few of the shortest periodic orbits is sufficient for an accurate estimate of $k_c(m/n)$ and $K(\nu)$ at any value of the argument. Our result represents an extension and application of the method of modular smoothing introduced in the references [2].

The most studied example of area-preserving twist maps is the Taylor-Chirikov standard map (SM), given by the following equations:

$$p_{i+1}=p_i+\frac{k}{2\pi}\sin(2\pi\theta_i), \tag{1}$$

$$\theta_{i+1}=\theta_i+p_{i+1}, \tag{2}$$

where k is a parameter. Our results will be illustrated using this system. Furthermore, the fractal diagram and the critical function are defined using only rotational orbits, that is, the orbits that go around the cylinder, so we shall be concerned only with such orbits.

The fractal diagram of the SM is defined as follows [3]. The standard map has a stable elliptic periodic orbit for any value of the rational frequency m/n and for a sufficiently small value of the parameter k. At some value of the parameter, which depends on the frequency m/n, the elliptic orbit bifurcates into an unstable reflection-hyperbolic orbit. The critical value of the parameter, at which the bifurcation occurs, as a function of the rational frequency is the fractal diagram $k_c(m/n)$. It is illustrated in Fig. 1. The stability of periodic orbits is usually described by its residue, which is defined by the following equation:

$$R(m/n;k)=\{2-\mathrm{Tr}[M(m/n;k)]\}/4, \tag{3}$$

where $M(m/n;k)$ is a product of Jacobian matrices of the map at all $n-1$ distinct points of the periodic orbit. A periodic orbit is elliptic if $0<R(m/n;k)<1$ and reflection hyperbolic if $R(m/n;k)>1$. Thus, the fractal diagram for the standard map can be defined by the following condition: $R(m/n,k_c(m/n))=1$.

Orbits with irrational frequencies are quasiperiodic. Depending on the value of the parameter, a quasiperiodic orbit with frequency ν can either fill an invariant circle in the phase space or an invariant Cantor subset of a circle. The value of the critical function $K(\nu)$ at an irrational frequency ν is the smallest value of the parameter k at which there is no invariant circle with frequency ν [4]. $K(\nu)$ is defined to be equal to zero at rational values of the argument. There is a relation, based on Greene conjecture [5], between the fractal diagram and the critical function. It is given by the following equation:

$$\lim_{m_i/n_i\to\nu} k_c(m/n)=K(\nu), \tag{4}$$

where m_i/n_i is the sequence of successive continued fraction convergents to the irrational ν.

The main problem with most of the existing techniques for calculations of the fractal diagram and the critical function is the local character. The computations have to be done for each relevant frequency again and independently from the calculations for other frequencies. For example, the calculations of the fractal diagram require first the calculations of the periodic orbits, and then their residues. Each periodic orbit has to be calculated separately and with no use of the knowledge about other distant orbits. The calculations of periodic orbits are usually based on some form of the Newton method and are relatively easy only in the case of the orbits with small periods [6]. The calculations of long periodic orbits close to the bifurcation points are quite time consuming and require special techniques to determine the good initial guess for the Newton method [7]. For example, although renormalization-group methods have produced some strong results, they are still local, and difficult to apply on realistic

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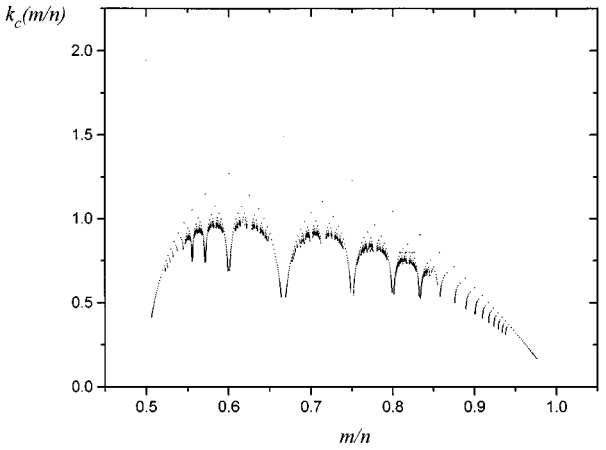

FIG. 1. Fractal diagram for the standard map.

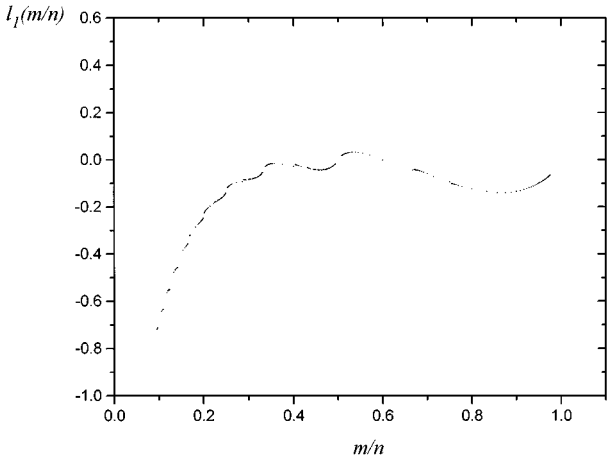

FIG. 2. The function $l_1(m/n)$ for the standard map.

systems [8,9]. However, a few years ago a method of modular smoothing for an efficient and relatively accurate calculation of fractal objects in Hamiltonian mechanics has been reported [2]. The method is based on the transformation properties of these objects under the action of the unimodular group on the frequencies. It has been applied on the critical functions directly and on the KAM tori. The purpose of this paper is to show that the method can be applied on the fractal diagrams. The main practical result is that a knowledge of $k_c(m/n)$ for just a few m/n with low n is sufficient for relatively accurate computation of all the fractal diagram and thus the critical function in the relevant problem, with almost no additional computations.

II. MODULAR SMOOTHING OF THE FRACTAL DIAGRAM

Detailed analyses of a perturbation expansion shows that the transformation properties of, for example, $K(\nu)$, can be described by a sequence of successively smoother functions $L_i(\nu)$ of the critical function and its transformations, with $L_0(\nu)=\ln K(\nu)$. The method is based on the cancellation of the singularities in $L_i(\nu)$ and $L_i(M\nu)$, where M is an element of the unimodular group

$$M\nu=\frac{a\nu+b}{c\nu+d}, \tag{5}$$

with a,b,c, and d integers satisfying $|ad-bc|=1$.

The function L_0 has infinite singularities at all rationals, but the function L_1 defined for the standard map by the formula

$$L_1(\nu)=L_0(\nu)-\nu L_0(\nu^{-1}) \tag{6}$$

is continuous everywhere and bounded (except at zero and infinity). Notice that the critical function of the standard map is invariant under integer translations of the frequency, so that $L_0(\nu)=L_0(\nu-\{\nu\})$ for any $\nu\in\mathbb{R}$. Here $\{\nu\}$ means the integer part of ν. The fractal diagram $k_c(m/n)$ is also invariant under integer translations of the rational argument m/n.

The function

$$L_2(\nu)=(\nu+1)L_1(\nu+1)-\nu L_1(\nu) \tag{7}$$

seems to be continuously differentiable, and so on. Furthermore, values of the functions $L_i, i>0$ at rationals can be calculated using perturbation expansion of a finite order. For example, for $L_1(m/n)$ we need perturbation expansion of the order n. A smooth interpolation L_{1a} through only a few points $L_1(m/n)$ then allows one to reconstruct the original fractal function with a bounded maximal error of just a few percent.

The method of modular smoothing enables one to approximate a discontinuous fractal function $K(\nu)$ by a smooth function, for example L_{1a}. It would be useful to be able to do the same also with the fractal diagram, which is a function of a similar complexity as the critical function. To this end we examined transformation properties of the fractal diagram $k_c(m/n)$ under the action of the generators of the unimodular group. By analogy with the situation in the case of the critical function we expect to be able to express these transformation properties via continuous and smooth functions. Notice that the asymptotic relation (4) between $k_c(m/n)$ and $K(\nu)$ does not imply that $k_c(m/n)$ must have the same properties as $K(\nu)$. However, guided by the definition of the function L_1, and the asymptotic relation (4), we defined, and calculated numerically, a new function $l_1(m/n)$ as follows:

$$l_1(m/n)=l_0(m/n)-\frac{m}{n}l_0(n/m), \tag{8}$$

where $l_0=\ln k_c(m/n)$.

The function $l_1(m/n)$ is presented in Fig. 2. It is obviously a continuous function, and can be extended by continuity onto the irrational frequencies. This is our main result.

Remarkably, l_1 coincides with the function L_1, defined via the critical function $K(\nu)$. However, we shall keep different notation in order to distinguish between $l_1(m/n)$, which is defined via periodic orbits and the fractal diagram, and $L_1(\nu)$, which is defined via the quasiperiodic orbits and the critical function. Also, we shall use $L_1(m/n)$ to denote the values of the function L_1 at rationals that are calculated by the perturbation expansion. The transformation properties of l_1 under the generators of the unimodular group are the same as for L_1, and are given by the following relations:

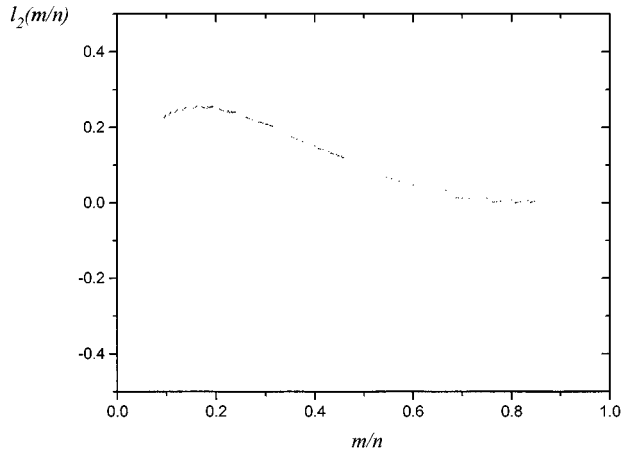

FIG. 3. The function $l_2(m/n)$ for the standard map.

$$l_1(1/\nu) = -|1/\nu| l_1(\nu), \tag{9}$$

$$l_1(-\nu) = l_1(\nu), \tag{10}$$

and

$$l_1(\nu+1) = \frac{l_2(\nu) + \nu l_1(\nu)}{\nu+1}. \tag{11}$$

The first two relations are simple consequences of the definition (8), but the third one contains a newly defined function $l_2(\nu)$, which describes the transformation of the function l_1 under the unit translation.

The function $l_2(m/n)$, obtained from numerical calculations of $k_c(m/n)$,$k_c(m/n+1)$ and $k_c[n/(m+n)]$ is presented in Fig. 3. We expect the function l_2 to be continuously differentiable, but there are regions in the numerical l_2, notably around 1/5, that raise doubts about such conclusion. On the other hand, these are the regions where the relative error in the numerical computations of $k_c(m/n)$ is largest, and the most evident in the values of l_2. In order to study these regions more carefully we shall first briefly describe the numerical method that we used for the calculations of $k_c(m/n)$, and point out the main problems in its application.

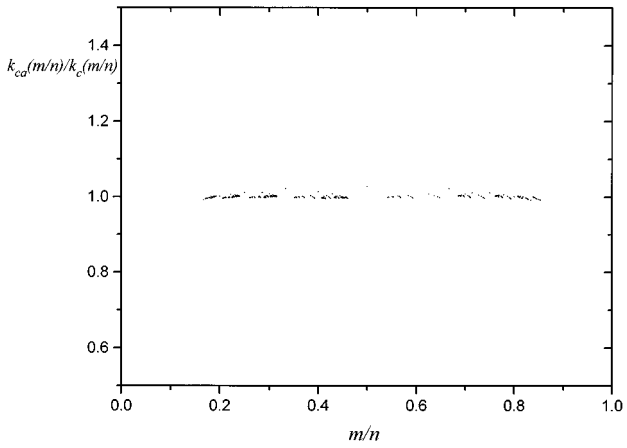

FIG. 5. The figure gives the ratio $k_{ca}(m/n)/k_c(m/n)$ for a sample set of all rationals with up to first five continued fraction coefficients ranging from 1 to 5.

The calculations of $k_c(m/n)$ are based on the calculations of the corresponding periodic orbits and checking if the orbit is subcritical $[R(m/n;k)<1]$ or supercritical $[R(m/n;k)>1]$. The only problem here is the calculation of long periodic orbits close to the bifurcation point. The periodic orbits are extremal points of an action functional [1]. For the calculation of the periodic orbits we used a stable method, based on the Greene-function approach, for solving difference equations, which come as the equations of the Newton method for finding zero of the action functional. The method was developed by Cheng, Menstel, and Percival [7], and there is a numerical package available [10]. As pointed out by the authors, and is always the case with a Newton method, success of the algorithm depends crucially on a good initial guess for the orbit (long near critical orbits are best approached by supercritical orbits). On the other hand, an unstable m/n orbit with a large period is close to an $m'/n' \approx m/n$ orbit with a short period that might be stable. In

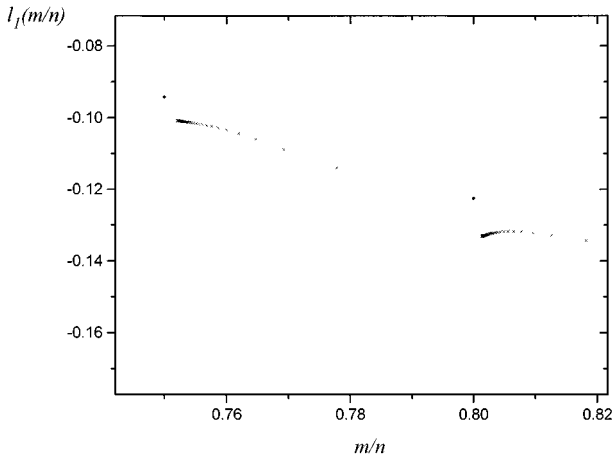

FIG. 4. The figure illustrates the significance of numerical errors in the calculations of $k_c(m/n)$ for the properties of numerically calculated $l_1(m/n)$. Points $l_1([0,a_1,a_2])$ are denoted by dots and the points $l_1([0,a_1,a_2,a_3]),a_3 \gg 1$ are denoted by crosses.

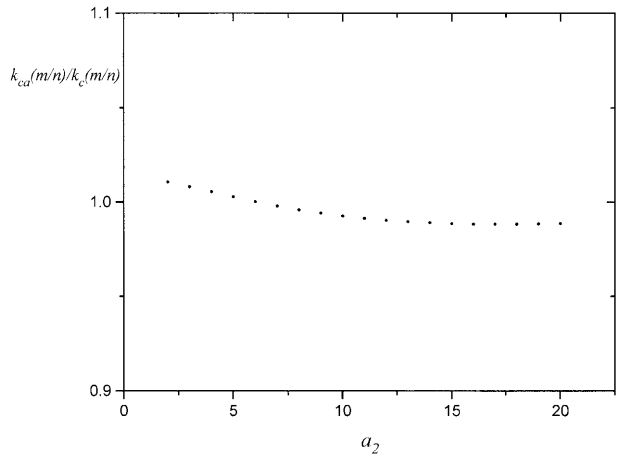

FIG. 6. The figure illustrates the ratio $k_{ca}(m/n)/k_c(m/n)$ for a sequence of rationals $m/n=\{0,3,a_2\},a_2=2,3,\ldots,20$ approaching $m/n=1/3$, vs the coefficient a_2. The figure strongly indicates that the errors in our approximate k_{ca} are bounded.

the numerical calculations it is very difficult to distinguish the two orbits. Thus it is likely that the absolute value of $R(m/n;k)$ might be underestimated, leading to a larger value of the numerical $k_c(m/n)$ than the correct one. This gives systematically larger values of the numerical $l_2(m/n)$ as well. This type of error can be illustrated also for the function $l_1(m/n)$, with a special choice of the rationals m/n. For example, the numerical $l_1(m/n)$ on the numbers m/n with the continued fraction expansion of a form $[0,a_1, a_2,a_3], a_3 \gg 1$, in an interval around 4/5, is illustrated in Fig. 4. It is quite different from the one illustrated in Fig. 2. This is obviously due to the numerical errors in the calculations of $k_c(m_i/n_i)$ on such rationals. The error is the most significant in the calculations of $k_c(m/n)$ for an m/n orbit that is close to $m'/n'=n/m-\{n/m\}$ orbit, with $k_c(m/n)$ considerably smaller then $k_c(m'/n')$. In conclusion, based on analytical reasons [2], numerical calculations for other systems [2], and parts of l_2 where it looks smooth, we believe that it is indeed a smooth function for all values of the argument.

III. TEST OF THE METHOD

We have tested the method using the standard map and an approximation of the function L_2, which was published in the second part of Ref. [2]. The approximation L_{2a} was obtained, as described in [2], by interpolating through only a few points $L_2(m/n)$ with small n. The algorithm for reconstructing the approximate fractal diagram from the approximate L_{2a} function is quite analogous to the algorithm, described in [2], for reconstructing the approximate critical function from the same L_{2a}. In the case of the critical function, defined on the irrationals, the approximation $K_a(\nu)$ at any noble ν is related to $K(\gamma), \gamma=\{0,1,1,\dots\}$ by a finite number of inversions and integer translations, using L_{2a}. In the case of the fractal diagram the approximation $k_{ca}(m/n)$ at any m/n is related to $k_c(1)=4$ by a finite number of inversions and integer translations, using the same L_{2a}. The algorithm was describe in [2], so we shall not duplicate the description here. The ratios $k_{ca}(m/n)/k_c(m/n)$ of the approximate fractal diagram $k_{ca}(m/n)$, obtained using its transformation properties approximated by L_{2a}, versus numerically calculated $k_c(m/n)$ are presented in Figs. 5 and 6. In Fig. 5 the set of frequencies m/n is chosen so as to illustrate the general behavior of the approximation. Let us point out that the calculations of $k_{ca}(m/n)$ at all points in the figure, using the reconstruction algorithm, take only a couple of seconds. In Fig. 6 we give the ratio on a sequence of rationals $m/n=\{0,3,a_2\}, a_2=2,3,\dots,20$, approaching a typical low-order resonance, namely, the resonance at 1/3, in order to demonstrate that errors are bounded. The figures illustrate that the errors are indeed quite small and bounded.

IV. SUMMARY

We shall now outline the procedure for the calculations of the boundaries of stability in a two degrees of freedom system, based on results reported in this paper. First, the system has to be reduced to a one-parameter family of systems with one and a half degrees of freedom. Then, one has to obtain the values of $k_c(m/n)$ for just a few of the shortest periodic orbits. Calculations of such orbits do not represent serious problems in any of the existing techniques. The few points $k_c(m/n)$ are used to calculate a few values of one of the $l_i, i>0$ functions. The values $l_i(m/n)$ serve as a skeleton for a smooth approximation $L_{ia}(\nu)$, which is than used to reconstruct the approximate fractal diagram $k_c(m/n)$ and the approximate critical function $K(\nu)$.

The functions $L_i(\nu)$ will be different for different systems, but we believe that the existence of these functions and the outlined procedure are general for a wide class of nondegenerate Hamiltonian systems with two degrees of freedom. Obviously, the procedure has to be applied and tested on some realistic systems.

The main practical use of our results is that, in realistic systems, it might be easier to use numerical computation of just a few short periodic orbits, in order to calculate the corresponding values of $l_1(m/n)$, than to perform the low-order perturbation expansion, which is needed for the calculations of $L_1(m/n)$ at these points. It should even be possible to apply the method to the data obtained directly from experiments, in which it is possible to detect the change in the stability of a few short periodic orbits.

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